Parametric Curves

(Section 10.1)

Unifies polar and rectangular coordinates

A parametric curve is a function from the real numbers to 2 (or more!) dimensions.

We write a parametric curve in 2 dimensions as

$$\boxed{f(t) = \langle x(t), y(t) \rangle}$$

for a real variable t.

Graphs: The graph of a parametric curve is the graph of all outputs of the curve: you don't plot the t values.

Example 1: (circle)

The equation $x^2+y^2=1$ does not yield the graph of a function.

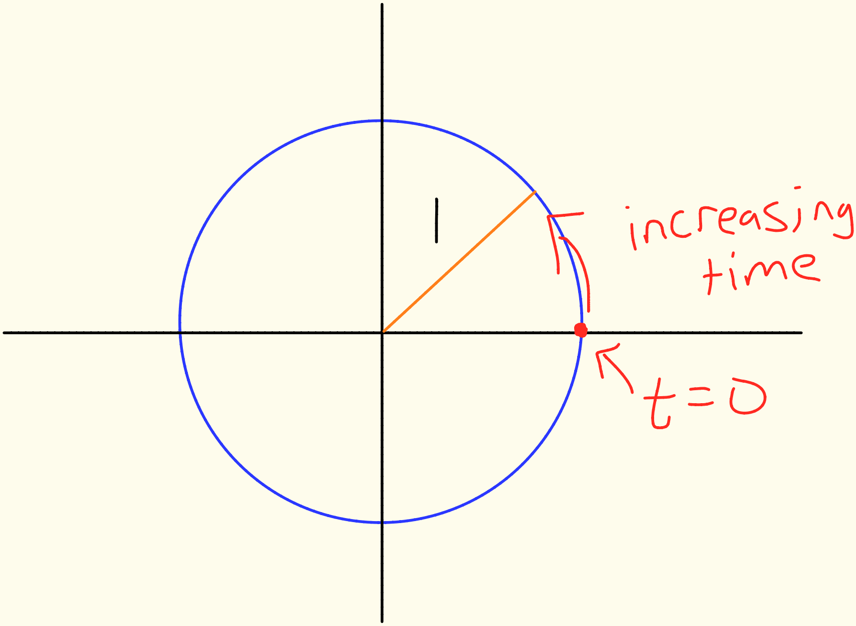
However, we can find a parametric expression that graphs this curve.

An example would be

$$\boxed{f(t)=\langle\cos(t),\sin(t)\rangle}$$

for $0\leq t\leq 2\pi$.

Graph:



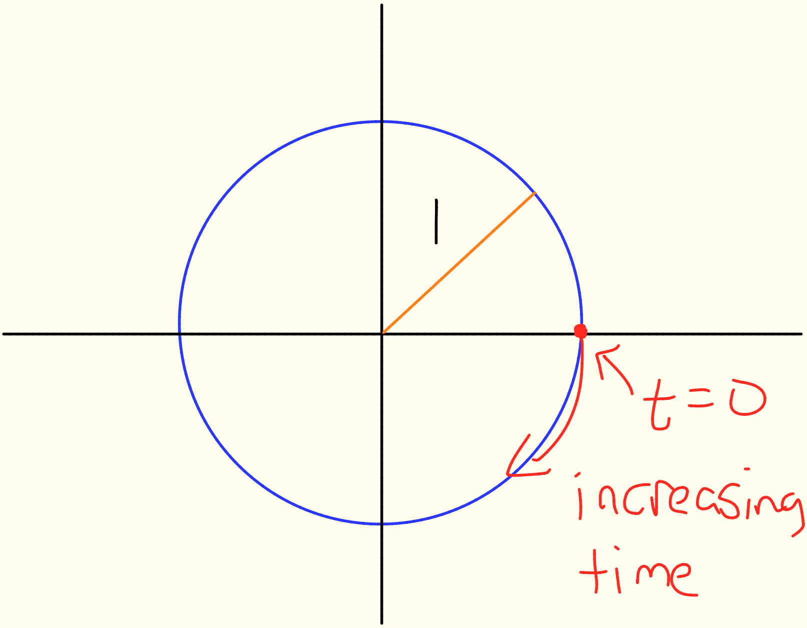
$$f(t) = \langle \cos(t), \sin(t) \rangle$$

The parameterization

$$g(t) = \langle \cos(t), -\sin(t) \rangle$$

for $0 \le t \le 2\pi$

gives the same graph traced in a different direction.



Some other parameterizations for the same curve:

$$\langle -\cos(t), \sin(t) \rangle$$

$$\langle -\cos(t), -\sin(t) \rangle$$

$$\langle \sin(t), \cos(t) \rangle$$

$$\langle \sin(16t), \cos(16t) \rangle$$

There are infinitely many such parameterizations.

Example 2: (any function)

Given a cartesian function $y = f(x)$, we can parameterize as

$$\boxed{g(t) = \langle t, f(t) \rangle}.$$

If $y = x$,

$$g(t) = \langle t, t \rangle$$

$$f(x) = \cos(\ln(2^{x+1}))$$

parameterization

$$g(t) = \langle t, \cos(\ln(2^{t+1})) \rangle$$

Example 3: (parametric to Cartesian)

Given

$$f(t) = \langle 11 - 3t^2, 5 - 8t^2 \rangle,$$

find a cartesian equation for the curve.

$$x(t) = 11 - 3t^2 = x$$
$$y(t) = 5 - 8t^2 = y$$

Pick one equation, solve for t^2.

$x = 11 - 3t^2$, so

$x - 11 = -3t^2$

$t^2 = \frac{11-x}{3}$

Substitute into

$y = 5 - 8t^2$

$= 5 - 8\left(\frac{11-x}{3}\right)$

$= \boxed{\frac{8x}{3} - \frac{73}{3}}$

which is the equation of a line! $(x \leq 11)$

Example 4: (ellipse)

Given

$$\frac{x^2}{400} + \frac{y^2}{(170)^2} = 1,$$

find a parameterization.

$$\boxed{f(t) = \langle 20\cos(t), 170\sin(t) \rangle}$$

$$(0 \le t \le 2\pi)$$

Check!

$$\frac{(20\cos(t))^2}{400} + \frac{(170\sin(t))^2}{(170)^2}$$

$$= \frac{\cancel{400}\cos^2(t)}{\cancel{400}} + \frac{\cancel{170^2}\sin^2(t)}{\cancel{170^2}}$$

$$= \cos^2(t) + \sin^2(t)$$

$$= 1$$

Example 5: (any polar curve)

Given a polar curve $\boxed{r = f(\theta)}$, use

$$x = r\cos(\theta) = f(\theta)\cos(\theta)$$

$$y = r\sin(\theta) = f(\theta)\sin(\theta)$$

Then a parameterization is

$$\boxed{g(\theta) = \langle f(\theta)\cos(\theta), f(\theta)\sin(\theta)\rangle}$$

You can substitute t for θ if you like!

If $r=1$,

$$g(\theta)=\langle \cos(\theta), \sin(\theta)\rangle.$$

If $r=\sin(\theta)+5$,

$$g(\theta)=\langle (\sin(\theta)+5)\cos(\theta), (\sin(\theta)+5)\sin(\theta)\rangle$$

Example 6: (logarithms)

If

$$f(t) = \left\langle \ln(2t), (\ln(t))^2 \right\rangle,$$

find a cartesian equation for the curve.

$$x = x(t) = \ln(2t)$$

$$= \ln(2) + \ln(t)$$

$$\ln(t) = x - \ln(2).$$

Substituting into

$$y = y(t)$$
$$= (\ln(t))^2$$
$$= (x - \ln(2))^2$$
$$= \boxed{x^2 - 2\ln(2)x + (\ln(2))^2}$$

(a quadratic)

In Webwork Question #4:

there is an equation in
the book that tells you
how all such curves look.

(Example 7, 10.1)

Tangent Lines to Parametric Curves

(Section 10.2)

This couldn't be easier:

$$f(t) = \langle x(t), y(t) \rangle,$$

assuming x, y differentiable

Assume that y can be written as a function of x (at least on some interval).

By the chain rule,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Dividing by $\frac{dx}{dt}$,

$$\boxed{\frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{dy}{dx}}$$

Written another way,

$$\boxed{\frac{dy}{dx} = \frac{y'(t)}{x'(t)}}$$

Except: If $\boxed{x'(t_0) = 0}$ (and $y'(t_0)$ doesn't cancel it out), then you have a vertical tangent at $t = t_0$.

If $\boxed{y'(t_0) = 0}$ (and $x'(t_0)$ doesn't cancel it out), then you have a horizontal tangent at $t = t_0$.

Example 1:

Let $f(t) = \langle t^t, \arcsin(t^2) \rangle$.

Find the equation of the tangent line at $t = \frac{1}{\sqrt{2}}$

$x(t) = t^t$

$y(t) = \arcsin(t^2)$.

At $t = 1$

$x\left(\frac{1}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{\sqrt{2}}}$

$y\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{6}$

$$x(t) = t^t = e^{\ln(t^t)}$$

$$= e^{t\ln(t)}$$

$$x'(t) = \underbrace{e^{t\ln(t)} \cdot \frac{d}{dt}(t\ln(t))}_{\text{chain rule}}$$

$$= e^{t\ln(t)}\left(\underbrace{t \cdot \frac{1}{t} + \ln(t) \cdot 1}_{\text{product rute}}\right)$$

$$= e^{t\ln(t)}(1 + \ln(t))$$

$$= t^t(1 + \ln(t))$$

$$y(t) = \arcsin(t^2)$$

$$y'(t) = 2t\left(\frac{1}{\sqrt{1-(t^2)^2}}\right)$$

$$= \frac{2t}{\sqrt{1-t^4}}$$

Equation of tangent line at $t = t_0$:

$$\frac{y'(t_0)}{x'(t_0)}\left(\boxed{x} - x(t_0)\right) = \boxed{y} - y(t_0)$$

- or -

$$\frac{y'(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{x - x(t_0)}$$

Circled functions = your parameterization

boxed = just the variables x & y.

We know: $t_0 = \frac{1}{\sqrt{2}}$

$$x\left(\frac{1}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{\sqrt{2}}}$$

$$y\left(\frac{1}{\sqrt{2}}\right) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$x'\left(\frac{1}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{\sqrt{2}}}\left(1 + \ln\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$y'\left(\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} \cdot \frac{1}{\sqrt{1 - \frac{1}{4}}}$$

$$= \frac{4}{\sqrt{6}}$$

Plug everything in:

$$\boxed{\frac{\left(\boxed{y} - \frac{\pi}{6}\right)}{\left(\boxed{x} - \left(\frac{1}{\sqrt{2}}\right)^{1/\sqrt{2}}\right)} = \frac{\left(4/\sqrt{6}\right)}{\left(\left(\frac{1}{\sqrt{2}}\right)^{1/\sqrt{2}}\left(1+\ln\left(\frac{1}{\sqrt{2}}\right)\right)\right)}}$$

Example 2:

If $f(t) = \langle \arctan(t), \cos(\pi t) \rangle$,

find the equation of the tangent line at the point

$$\left(\frac{\pi}{4}, -1\right).$$

Good: you don't need to find a point on the line; it is given!

Bad: What is t_0?

$$x(t) = \arctan(t)$$

$$x'(t) = \frac{1}{1+t^2}$$

$$y(t) = \cos(\pi t)$$

$$y'(t) = -\pi \sin(\pi t).$$

What point do we plug in for t? We know

$$x(t_0) = \frac{\pi}{4} = \arctan(t_0)$$

$t_0 = 1$ - Check:

$$\cos(\pi) = -1 = y(t_0) \checkmark$$

$$t_0 = 1$$

$$x'(t_0) = \frac{1}{1+1} = \frac{1}{2}$$

$$y'(t_0) = -\pi \sin(\pi)$$

$$= 0$$

This tells you the tangent line is horizontal.

Equation: $\boxed{y = -1}$

Example 3: Find where the tangent line is horizontal if

$$f(t) = \langle t^3 - 4t^2 - 2t,\ t^2 - 4t + 1 \rangle.$$

Horizontal: $y' = 0$

$$y'(t) = 2t - 4 = 0$$

$$t = 2$$

Check: $x'(2) \neq 0$.

$$x(t) = t^3 - 4t^2 - 2t$$

$$x'(t) = 3t^2 - 8t - 2$$

$$x'(2) = 12 - 16 - 2$$

$$= -6 \neq 0$$

So there is a horizontal tangent at $t = 2$ corresponding to the point

$$(x(2), y(2))$$

$$= \boxed{(-12, -3)}$$

Example 4: Find all points where

$$f(t) = \langle t^3 - 4t^2 - 2t,\ t^2 - 4t + 1 \rangle$$

has a vertical tangent.

vertical: $x'(t) = 0$

$$x'(t) = 3t^2 - 8t - 2 = 0$$

quadratic formula

$$t = \frac{8 \pm \sqrt{64 + 24}}{6} = \frac{8 \pm \sqrt{88}}{6}$$

vertical tangents when

$$t = \frac{8 \pm \sqrt{88}}{6}, \text{ points}$$

$$\boxed{\left(x\left(\frac{8 \pm \sqrt{88}}{6}\right), y\left(\frac{8 \pm \sqrt{88}}{6}\right)\right)}$$

Whatever these numbers are

2^{nd} derivatives

If $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$,

replace y with dy/dx

then

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

Example 5:

If $f(t) = \langle 3t^2+5,\ 2^t \rangle$,

find $\dfrac{d^2y}{dx^2}$ at $t=0$.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\ln(2)\,2^t}{6t}$$

Using $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}$ . . .

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{\ln(2)2^t}{6t}\right)}{6t}$$

$$= \frac{\left(\frac{6t(\ln(2))^2 \cdot 2^t - 6\ln(2)2^t}{(6t)^2}\right)}{6t}$$

$$= \frac{6t(\ln(2))^2 \cdot 2^t - 6\ln(2)2^t}{(6t)^3}$$

plug in $t=0$!

We get $\dfrac{-6\ln(2)}{0}$,

$\boxed{\text{vertical tangent}}$.